Monochromatic waves induced by large-scale parametric forcing

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We study the formation and stability of monochromatic waves induced by large-scale modulations in the framework of the complex Ginzburg-Landau equation with parametric nonresonant forcing dependent on the spatial coordinate. In the limiting case of forcing with very large characteristic length scale, analytical solutions for the equation are found and conditions of their existence are outlined. Stability analysis indicates that the interval of existence of a monochromatic wave can contain a subinterval where the wave is stable. We discuss potential applications of the model in rheology, fluid dynamics, and optics.

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The complex Ginzburg-Landau equation (CGLE) is a generic mathematical model describing nonequilibrium dynamics of spatially extended systems, including spontaneous wave modulation, spatiotemporal chaos, intermittency, oscillatory instabilities, and others [1,2]. This equation is invariant with respect to gauge transformation, spatial reflections, and spatiotemporal translations. Coherent and chaotic solutions to this equation exhibit rich dynamical properties [1].

Many natural and artificial systems that manifest pattern-forming behavior are multiscale and inhomogeneous [3]. The heterogeneities induce spatial and temporal modulations (either regular or random) that influence significantly the nonequilibrium dynamics [4–9]. In some cases, modulations result in a break of the gauge invariance of the pattern-forming system. This happens, for instance, when a control parameter of the system is modulated by a periodic perturbation with a wavelength close to resonance. This modulation creates an additional term in the CGLE that violates the gauge invariance of the model equation [6] and leads to a number of interesting dynamic effects; see [8] for a comprehensive review of the studies.

The subject of our research is a "nonintrusive" forcing that, on one hand, preserves the internal symmetries of the system and the model equation and, on the other hand, accurately captures the effect of heterogeneities on the nonlinear dynamics. Such scrupulous consideration is especially important for modeling dynamics of complex multiscale phenomena that are hardly accessible to first-principles analysis (see Ref. [3] and references therein).

Recent work [3] has performed the first systematic analytical and numerical study of the influence of parametric nonresonant forcing periodic in space and time on nonequilibrium dynamics of wave patterns. A number of interesting properties of the multiscale dynamics were found, and a need of further investigations was indicated [3]. It was shown, for instance, that the forcing results in the occurrence of traveling waves with new dispersion properties and may completely suppress the development of an intermittent chaos [3].

Here, we consider the influence of modulations on the formation of monochromatic wave patterns in the frames of CGLE with parametric nonresonant forcing. Conditions of existence of these solutions are outlined and their stability is analyzed. It is shown that the interval of existence can contain a stability subinterval. In the case of spatially periodic forcing the monochromatic wave is quasiperiodic rather than periodic.

The complex Ginzburg-Landau equation can be applied for a description of two types of oscillatory instabilities: the *long-wave* instability developing when the neutral curve of the control parameter $\mu=\mu(k)$ has its minimum $\mu=\mu_0$ at wave number $k=0$, and the *short-wave* instability developing when the neutral curve $\mu=\mu(k)$ has its minimum at a finite wave number $k=k_0$.

The long-wave oscillatory instability occurs, for instance, in a reaction-diffusion system when a pure reaction system described by ordinary differential equations (ODEs) is subject to Hopf bifurcation at a certain value of the control parameter $\mu=\mu_0$. Assuming that the control parameter μ is a slow function of space coordinate x,

$$\mu=\mu_0+\epsilon^2\mu_2(x_1), \quad x_1=\epsilon x, \quad \epsilon \ll 1,$$

employing the eigenvectors of the ODE problem at $\mu=\mu_0$ with a slowly changing amplitude function $A(x_1,t_2)$, $t_2=\epsilon^2 t$, and using the multiscale approach, one can reduce the problem (to the leading order) to a complex Ginzburg-Landau equation,

$$\frac{\partial A}{\partial t_2}=\lambda_2(x_1)A+D\frac{\partial^2 A}{\partial x_1^2}-\kappa|A|^2A. \tag{1}$$

Here, $\lambda_2(x_1)=\mu_2(x_1)(d\lambda/d\mu)_{\mu_0}$ is the spatially modulated growth rate coefficient, whereas complex coefficients D and κ are equal to those in the homogeneous case with $\mu=\mu_0$.

The second type of instabilities governed by the complex Ginzburg-Landau equation is the short-wave oscillatory instability developing when the minimum of the neutral curve $\mu=\mu(k)$ is situated at a finite wave number $k=k_0\neq 0$ [10,11]. For the short-wave instability the amplitude function A has the meaning of an envelope function of a wave packet with a base wave number k_0. If the group velocity of waves at the critical point $\mu=\mu_0$, $k=k_0$ is small because of some physical reasons, $v=\epsilon v_1$, the leading-order evolution of the wave packet takes place on the time scale $t_2=\epsilon^2 t$ and is governed by the equation

$$\frac{\partial A}{\partial t_2} + v_1 \frac{\partial A}{\partial x_1} = \lambda_2(x_1)A + D\frac{\partial^2 A}{\partial x_1^2} - \kappa |A|^2 A. \quad (2)$$

Equation (1) can be considered as a particular case of Eq. (2). Therefore, in the foregoing discussion we do not distinguish between these two physically distinct cases. By rescaling and redefining variables, Eq. (2) can be represented in the form

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial x} = [f(x) + ig(x)]A + (1 + i\alpha)A_{xx} - (1 + i\beta)|A|^2 A. \quad (3)$$

Here, $f(x)$ and $g(x)$ characterize the influence of external spatial inhomogeneity on the growth rate and oscillation frequency correspondingly, and coefficients α and β that describe the dispersion and nonlinear frequency shift are standard for the Ginzburg-Landau equation. For $f=1$, $g=\alpha=\beta=v=0$, Eq. (3) is reduced to the real Ginzburg-Landau equation. For $g=1$, $f=v=0$ and α, $\beta\to\infty$, Eq. (3) is reduced to the nonlinear Schrödinger equation.

With $A(x,t)=R(x,t)\exp[i\Theta(x,t)]$ we obtain the following system of equations:

$$R_t + vR_x = (f - R^2 - \Theta_x^2)R - \alpha(2R_x\Theta_x + R\Theta_{xx}) + R_{xx},$$

$$R(\Theta_t + v\Theta_x) = (g - \alpha\Theta_x^2 - \beta R^2)R + 2R_x\Theta_x + R\Theta_{xx} + \alpha R_{xx}. \quad (4)$$

In the present Brief Report, we consider modulations with the spatial scale large compared to the characteristic length scale of the complex Ginzburg-Landau equation, i.e., $f=f(kx)$, $g=g(kx)$, where $k \ll 1$. The analysis carried out in [3] has revealed a class of solutions in the form of spatially modulated and temporally monochromatic waves with

$$R = R(x), \quad \Theta = \theta(x) - \omega t, \quad (5)$$

where ω is being a constant. Here, this class of solutions is studied for $v \neq 0$ and for both periodic and nonperiodic forcing functions $f=f(kx)$ and $g=g(kx)$.

Defining $Q=\theta_x$ (local wave number), assuming $R=R(kx)$, $Q=Q(kx)$, and introducing a rescaled spatial variable $X=kx$, we find that the monochromatic wave is governed by the following system of equations:

$$(f - R^2 - Q^2)R - k[vR_X + \alpha(2R_XQ + RQ_X)] + k^2R_{XX} = 0,$$

$$(g + \omega - vQ - \alpha Q^2 - \beta R^2)R + k(2R_XQ + RQ_X) + \alpha k^2 R_{XX} = 0. \quad (6)$$

We assume that in the foregoing discussion $\alpha \neq \beta$ (nonresonant case; see, for details, [3]).

The solution for Eq. (6) can be found as a power series in terms of small k,

$$R = R_0 + kR_1 + \cdots, \quad Q = Q_0 + kQ_1 + \cdots. \quad (7)$$

Due to the singular nature of Eq. (6), expansion (7) is actually an *outer* expansion. Focus our attention on the solutions that do not contain *internal layers* characterized by a rapid solution change.

To the zeroth order in k, we obtain from Eqs. (3)–(7) a system of algebraic equations,

$$(f - R_0^2 - Q_0^2)R_0 = 0, \quad (g + \omega - vQ_0 - \alpha Q_0^2 - \beta R_0^2)R_0 = 0.$$

Considering nontrivial solutions with $R_0 \neq 0$, we find that the wave amplitude R_0 is slaved to the local wave number Q_0 through the relation $R_0^2=f-Q_0^2$, whereas the wave number satisfies the quadratic equation

$$(\alpha - \beta)Q_0^2 + vQ_0 - \omega + \beta f - g = 0. \quad (8)$$

For a nonresonant case with $\alpha \neq \beta$, Eq. (8) has real solutions,

$$Q_0^{\pm} = \frac{-v \pm \sqrt{D(X)}}{2(\alpha - \beta)}, \quad (9)$$

if the value of

$$D(X) = v^2 + 4(\alpha - \beta)(\omega - \beta f + g) \quad (10)$$

is non-negative everywhere in the region. For given functions $f(X)$ and $g(X)$, the requirements $D(x)\geq 0$ and $R_0^2=f-Q_0^2>0$ impose certain restriction on possible values of frequency ω and thus outline the region of existence of the monochromatic wave.

In the case of $\alpha-\beta>0$, the value of D grows with ω; therefore, the condition $D(x)\geq 0$ is satisfied for $\omega \geq \omega_*$, where $\omega_*=\beta f-g-v^2/4(\alpha-\beta)$. At the point $\omega=\omega_*$ both branches of the solution in Eq. (9) merge at $Q_0=Q_*=-v/2(\alpha-\beta)$. If $f>v^2/4(\alpha-\beta)^2$, the branches $Q_0^{\pm}$ satisfy the condition $Q_0^2<f$ in the interval of frequencies $\omega_* \leq \omega < \omega_{\pm}$, where $\omega_{\pm}=\alpha f-g\pm v\sqrt{f}$. If $0<f<v^2/4(\alpha-\beta)^2$, only one of the branches $Q_0^{\pm}$ can satisfy that condition. In particular, if $v>0$, then the branch Q_0^{+} exists as $\omega_- < \omega < \omega_+$. For $v<0$ the branch Q_0^{-} exists as $\omega_+ < \omega < \omega_-$.

In the case of $\alpha-\beta<0$, the solutions exist in the region $\omega \leq \omega_*$. Similarly, for $f>v^2/4(\alpha-\beta)^2$ the branches $Q_0^{\pm}$ exist in the intervals $\omega_{\mp} < \omega \leq \omega_*$, while for $0<f<v^2/4(\alpha-\beta)^2$ the branch Q_0^{+} exists in the interval $\omega_- < \omega < \omega_+$, if $v<0$, and the branch Q_0^{-} exists in the interval $\omega_+ < \omega < \omega_-$, if $v>0$.

Recall that in the case of a nonmodulated growth rate parameter ($f=1$, $g=0$) the monochromatic wave with

$$R^2 = 1 - Q^2, \quad \omega = vQ + (\alpha - \beta)Q^2$$

is stable within the "Busse balloon" [3],

$$Q^2 < Q_m^2 = \frac{1 + \alpha\beta}{3 + \alpha\beta + 2\beta^2}. \quad (11)$$

The wave is unstable outside this interval because of the Eckhaus (long-wave phase-modulation) instability associated with the breaks of both translational invariance ($x \to x + C_1$) and gauge invariance ($\Theta \to \Theta + C_2$) of the original problem. The modulated forcing violates the translational invariance and conserves the gauge invariance.

For the sake of simplicity, we investigate the stability of the solution only in the case of $v=0$, $g(x)=0$, where

$$Q_0^2 = \frac{\omega - \beta f}{\alpha - \beta}, \quad R_0^2 = -\frac{\omega - \alpha f}{\alpha - \beta}. \tag{12}$$

The admissible values of ω are inside an interval $\omega_- < \omega < \omega_+$. On one border of this interval $Q_0 \to 0$, and on the other border $R_0 \to 0$. For each admissible value of ω there are two solutions with different signs of Q_0. The interval disappears when the conditions $Q_0^2 > 0$ and $R_0^2 > 0$ cannot be satisfied for any X.

Determine from Eq. (4) the first-order corrections R_1 and Q_1 for the monochromatic solution. With account for the relation $Q_0^2 = f - R_0^2$, Eq. (4) is transformed to

$$2R_0^2 R_1 + 2Q_0 R_0 Q_1 = -\alpha(2R_{0X}Q_0 + R_0 Q_{0X}),$$

$$2\beta R_0^2 R_1 + 2\alpha Q_0 R_0 Q_1 = 2R_{0X}Q_0 + R_0 Q_{0X},$$

and one derives that

$$R_1 = -\frac{1+\alpha^2}{2(\alpha-\beta)R_0^2}(2R_{0X}Q_0 + R_0 Q_{0X}),$$

$$Q_1 = \frac{1+\alpha\beta}{2(\alpha-\beta)Q_0 R_0}(2R_{0X}Q_0 + R_0 Q_{0X}).$$

Using variables $Q = \Theta_x$ and $X = kx$ (recall that $v = 0$ and $g = 0$), we find from the time-dependent problem (4) that

$$R_t = (f - R^2 - Q^2)R - k\alpha(2R_X Q + R Q_X) + k^2 R_{XX},$$

$$Q_t = -k(\alpha Q^2 + \beta R^2)_X + k^2(2R_X K/R + K_X)_X + k^3\alpha(R_{XX}/R)_X, \tag{13}$$

and linearizing Eq. (13) around the monochromatic solution $[R(X), Q(X)]$, we obtain the following system of equations for the disturbances $[\tilde{R}(X), \tilde{Q}(X)]\exp(\sigma t)$:

$$\sigma\tilde{R} = (f - 3R^2 - Q^2)\tilde{R} - 2QR\tilde{Q} - k\alpha(2R_X\tilde{Q} + 2\tilde{R}_X Q + \tilde{R}Q_X + R\tilde{Q}_X) + k^2\tilde{R}_{XX},$$

$$\sigma\tilde{Q} = -k(2\alpha Q\tilde{Q} + 2\beta R\tilde{R})_X + k^2(2\tilde{R}_X Q/R + 2R_X\tilde{Q}/R - 2R_X Q\tilde{R}/R^2 + \tilde{Q}_X)_X + \alpha k^3(\tilde{R}_{XX}/R - R_{XX}\tilde{R}/R^2)_X. \tag{14}$$

To perform a stability study, we apply power-law expansions in terms of k for Q, R, $\tilde{Q}$, $\tilde{R}$, and σ.

In the zeroth order,

$$\sigma_0\tilde{R}_0 = -2R_0^2\tilde{R}_0 - 2Q_0R_0\tilde{Q}_0, \quad \sigma_0\tilde{Q}_0 = 0.$$

For nondecaying disturbances,

$$\sigma_0 = 0, \quad \tilde{R}_0 = -Q_0\tilde{Q}_0/R_0. \tag{15}$$

In the first order, we find

$$2R_0^2\tilde{R}_1 + 2Q_0R_0\tilde{Q}_1 = -\sigma_1\tilde{R}_0 - (2Q_0Q_1 + 6R_0R_1)\tilde{R}_0 - 2(Q_0R_1 + Q_1R_0)\tilde{Q}_0 - \alpha(2R_{0X}\tilde{Q}_0 + 2\tilde{R}_{0X}Q_0 + \tilde{R}_0Q_{0X} + R_0\tilde{Q}_{0X}), \tag{16}$$

$$\sigma_1\tilde{Q}_0 = -(2\alpha Q_0\tilde{Q}_0 + 2\beta R_0\tilde{R}_0)_X. \tag{17}$$

Substituting relation (15) in Eq. (17), we find that

$$\tilde{Q}_0(X) = \frac{1}{Q_0(X)}\exp\left[-\frac{\sigma_1}{2(\alpha-\beta)}\int_0^X \frac{d\xi}{Q_0(\xi)}\right] \tag{18}$$

up to an arbitrary coefficient. As $Q_0(\xi)$ is real and does not change its sign, the integral $\int_0^X d\xi/Q_0(\xi)$ is real and nonzero. On the other hand, function (18) is bounded at $X \to \pm\infty$ only if σ_1 is purely imaginary, $\sigma_1 = -i\omega_1$.

Thus, we obtain a one-parametric family of perturbations,

$$\tilde{Q}_0(X) = \frac{1}{Q_0(X)}\exp[i\Phi_0(X)], \quad \tilde{R}_0(X) = -\frac{1}{R_0(X)}\exp[i\Phi_0(X)], \tag{19}$$

where

$$\Phi_0(X) = \frac{\omega_1}{2(\alpha-\beta)}\int_0^X \frac{d\xi}{Q_0(\xi)}. \tag{20}$$

As $\mathrm{Re}[\sigma_1] = 0$, we cannot estimate the growth or decay of the perturbations. An explicit expression for $\tilde{R}_1$ can be found from the algebraic equation (16); however, it is cumbersome, and it is not presented here.

The stability criterion is obtained in the second order from the equation for $\tilde{Q}_1$, which can be written in following form:

$$i\omega_1\tilde{Q}_1 + 2(\alpha-\beta)(Q_0\tilde{Q}_1)_X = \left(-\frac{\sigma_2}{Q_0} + F\right)\exp(i\Phi_0), \tag{21}$$

where F is a (cumbersome) expression containing the functions that have already been found. Substituting

$$\tilde{Q}_1 = \frac{\Psi}{Q_0}\exp(i\Phi_0),$$

we obtain the following equation for Ψ:

$$2(\alpha-\beta)\Psi_X = -\frac{\sigma_2}{Q_0} + F.$$

If the external modulation function $f(X)$ is periodic with the period L, then the solution is bounded at infinity and periodic with the period L under the condition

$$\int_0^L \left(-\frac{\sigma_2}{Q_0(\xi)} + F(\xi)\right)d\xi = 0.$$

Therefore,

$$\sigma_2 = \frac{\langle F\rangle}{\langle Q_0^{-1}\rangle},$$

where

$$\langle u\rangle \equiv \frac{1}{L}\int_0^L u(\xi)d\xi.$$

In the case of a nonperiodic external forcing, the variables are averaged over the entire region $-\infty < X < \infty$.

Substituting the explicit expression of F and collecting all the terms that are not derivatives of periodic functions and hence contribute to the integral, we find

$$\sigma_2 = \frac{\omega_1^2}{4(\alpha-\beta)^2}\frac{-(1+\alpha\beta)\langle Q_0^{-3}\rangle + 2(1+\beta^2)\langle Q_0^{-1}R_0^{-2}\rangle}{\langle Q_0^{-1}\rangle}. \tag{22}$$

Thus, the monochromatic wave is stable if for this wave,

$$\frac{\langle Q_0^{-1}R_0^{-2}\rangle}{\langle Q_0^{-3}\rangle} < \frac{1+\alpha\beta}{1+\beta^2}, \tag{23}$$

and it is unstable, otherwise. The functions $Q_0(X)$ and $R_0(X)$ are determined by formulas (12).

In the absence of the modulation ($f=1$) the Eckhaus instability criterion (11) is recovered from Eq. (23). On the border of the interval of admissible values of ω, where $R_0 \to 0$, the left-hand side of Eq. (23) diverges; hence, the monochromatic wave is unstable. On another border, where $Q_0 \to 0$, the left-hand side of Eq. (23) vanishes; hence, the monochromatic wave is stable for $1+\alpha\beta > 0$. The latter inequality coincides with the condition of the absence of a Benjamin-Feir instability for nonmodulated waves. Thus, if this inequality is satisfied, the interval of existence of a monochromatic wave always contains a subinterval where this wave is stable.

We have studied the structure and stability of wave patterns induced and influenced by modulations in the framework of the complex Ginzburg-Landau equation with parametric nonresonant forcing. The forcing is preserving gauge invariance of the system and is dependent on the spatial coordinate. It is found that in the case of forcing with a very large characteristic length scale, there exists an interval where the nonlinear solutions are monochromatic waves; and, furthermore, there exists a subinterval, where these waves are stable. For periodic forcing the monochromatic waves are quasiperiodic rather than periodic. In the absence of modulations, the stability conditions coincide with the criterion for the Eckhaus instability associated with the violation of gauge invariance. In the presence of forcing, despite the violation of translational invariance, the growth rate spectrum of long-wave disturbances is similar to that in the spatially homogeneous case. Instead of the wave number, which is not an adequate parameter in the absence of the translational invariance, the wave frequency can be used for the parametrization of the family of disturbances.

Some of potential applications of our model include thermal convection, suspensions of long rods in rheology, light propagation in rotating waveguide arrays in optics, and many others [3]. Detailed consideration of these applications is beyond the scope of our theoretical Brief Report and can be a subject for further research.

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